Global well-posedness and scattering for the defocusing, energy -critical, nonlinear Schrödinger equation in the exterior of a convex obstacle when $d = 4$

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May 15, 2012

Abstract: In this paper we prove that the energy - critical nonlinear Schrödinger equation in the domain exterior to a convex obstacle is globally well - posed and scattering for initial data having finite energy. To prove this we utilize frequency localized Morawetz estimates adapted to an exterior domain.

1 Introduction

In this paper we study the defocusing, energy - critical nonlinear Schrödinger equation

$$\begin{aligned} iu_t + \Delta u &= |u|^2 u, \\ u(0,x) &= u_0 \in \dot{H}_0^1(\Omega), \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{1.1}$$

where $\Omega = \mathbf{R}^4 \setminus \Sigma$ is an exterior domain, Σ is a compact, convex obstacle. We prove

Theorem 1.1 *For $u_0 \in \dot{H}_0^1(\Omega)$, $d = 4$, (1.1) is globally well - posed and scattering.*

As in the case when u solves $iu_t + \Delta u = |u|^2 u$ on $\mathbf{R}^4$, a solution to (1.1) conserves the quantities mass,

$$M(u(t)) = \int_\Omega |u(t,x)|^2 dx = M(u(0)), \tag{1.2}$$

and energy

$$E(u(t)) = \frac{1}{2}\int_\Omega |\nabla u(t,x)|^2 dx + \frac{1}{4}\int_\Omega |u(t,x)|^4 dx = E(u(0)). \tag{1.3}$$

(1.1) is called energy critical since in $\mathbf{R}^4$ the symmetry $u(t,x) \mapsto \frac{1}{\lambda}u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ maps solutions to solutions and preserves energy. Of course, in the case of an exterior domain this scaling symmetry does not map solutions of (1.1) to solutions of (1.1). However, this problem behaves like the energy critical problem in $\mathbf{R}^4$ in many respects.

For a domain exterior to a non - trapping obstacle [23] and [1] proved that the quintic problem $iu_t + \Delta u = |u|^4 u$ is globally well - posed and scattering for $E(u_0)$ sufficiently small, u satisfies Dirichlet or Neumann boundary conditions. [28] proved global well - posedness and scattering for the defocusing quintic problem when $d = 3$, u is radial, and the domain is the exterior of a unit ball. The results of [23] and [1] correspond to the results of [5] for the quintic problem when $d = 3$. Likewise, the techniques of [28] utilize the induction on energy technique used in [4] and [33].

Theorem 1.2 *Let $\Omega = \mathbf{R}^d \setminus \mathcal{K}$ be the exterior domain to a compact nontrapping obstacle with smooth boundary, and Δ the standard Laplace operator on Ω, subject to either Dirichlet or Neumann conditions. Suppose that $p > 2$ and $q < \infty$ satisfy*

$$\begin{aligned} &\frac{3}{p} + \frac{n}{q} \leq \frac{n}{2}, \qquad n = 2, \\ &\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, \qquad n \geq 3. \end{aligned} \tag{1.4}$$

Then for the solution $v = \exp(it\Delta)f$ to the Schrödinger equation

$$\begin{aligned} iv_t + \Delta v &= 0, \\ v(0,x) &= f, \\ v|_{\partial\Omega} = 0, \qquad &\textit{or} \qquad \partial_\nu v|_{\partial\Omega} = 0, \end{aligned} \tag{1.5}$$

the following estimates hold

$$\|v\|_{L_t^p([-T,T];L^q(\Omega))} \leq C\|f\|_{\dot{H}^s(\Omega)}, \tag{1.6}$$

provided that

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} - s. \tag{1.7}$$

Proof: See [1]. $\Box$

In dimensions $d \geq 4$ the Strichartz estimates of [1] are not sufficient to prove even small data global well - posedness of the energy - critical problem. Therefore, we will be content to consider the domain exterior to a convex obstacle, where we have an almost full range of Strichartz estimates.

Theorem 1.3 *Suppose $u(t,x)$ is a solution to the linear Schrödinger equation with Dirichlet boundary conditions*

$$\begin{aligned} iu_t + \Delta_D u &= F, \\ u(0,x) &= u_0, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{1.8}$$

A pair will be called admissible if $p > 2$ and

$$\frac{2}{p} = d(\frac{1}{2} - \frac{1}{q}). \tag{1.9}$$

For (p,q), $(\tilde{p},\tilde{q})$ admissible

$$\|u\|_{L_t^p(I;L^q(\Omega))} \lesssim_{p,\tilde{p}} \|u_0\|_{L^2(\Omega)} + \|F\|_{L_t^{\tilde{p}'}(I;L^{\tilde{q}'}(\Omega))}. \tag{1.10}$$

Proof: See [21] $\Box$.

This theorem automatically gives small energy global well - posedness and scattering for (1.1).

We will be able to prove theorem 1.1 by utilizing the frequency truncated Morawetz estimates used in the mass - critical problem (see [18], [17], [16], [19]) on $\mathbf{R}^d$. This technique was also used for the defocusing, energy - critical problem in $\mathbf{R}^d$, $d = 3, 4$. (See [40] and [25].) We will borrow terminology from [40] and [25] and deal with the rapid frequency cascade and the quasi - soliton solution separately. Due to lack of scale invariance and translation invariance we will not make a concentration compactness argument. Instead, we will use induction on energy. However, the arguments used are quite reminiscent of the arguments found in [18], [17], [16], [19], [40], and [25]. A quick glance at [40] and [25] will show that one might expect that the energy - critical problem in $\mathbf{R}^4 \setminus \Sigma$ is substantially easier than the energy - critical problem in $\mathbf{R}^3 \setminus \Sigma$. The energy - critical problem in $\mathbf{R}^3 \setminus \Sigma$ remains out of the reach of the techniques used in this paper.

Function Spaces

It will be convenient to utilize the function spaces which are a superposition of free solutions to the Schrodinger equation. See [27], [20] for more information.

Definition 1.1 *Let $1 \leq p < \infty$. Then $U^p_{\Delta_D}$ is an atomic space, where atoms are piecewise solutions to the linear equation $iu_t + \Delta_D u = 0$, where $\Delta_D = \Delta$ in the interior of Ω, $\Delta_D = 0$ on $\partial\Omega$.*

$$u = \sum_k 1_{[t_k,t_{k+1})} e^{it\Delta_D} u_k, \quad \sum_k \|u_k\|_{L^2}^p = 1. \tag{1.11}$$

For any function u,

$$\|u\|_{U^p_{\Delta_D}} = \inf\{\sum_\lambda |c_\lambda| : u = \sum_\lambda c_\lambda u_\lambda, u_\lambda \text{ are } U^p_{\Delta_D} \text{ atoms}\} \tag{1.12}$$

For any $1 \leq p < \infty$, $U^p_{\Delta_D} \subset L^\infty L^2$. Additionally, $U^p_{\Delta_D}$ functions are continuous except at countably many points and right continuous everywhere.

Definition 1.2 *Let $1 \leq p < \infty$. Then $V^p_{\Delta_D}$ is the space of right continuous functions $u \in L^\infty(L^2)$ such that*

$$\|v\|^p_{V^p_{\Delta_D}} = \|v\|^p_{L^\infty(L^2)} + \sup_{\{t_k\}\nearrow} \sum_k \|e^{-it_k\Delta_D} v(t_k) - e^{-it_{k+1}\Delta_D} v(t_{k+1})\|^p_{L^2}. \tag{1.13}$$

The supremum is taken over increasing sequences t_k.

Theorem 1.4 *The function spaces $U^p_{\Delta_D}$, $V^q_{\Delta_D}$ obey the embeddings*

$$U^p_{\Delta_D} \subset V^p_{\Delta_D} \subset U^q_{\Delta_D} \subset L^\infty(L^2), \quad p < q. \tag{1.14}$$

Let $DU^p_{\Delta_D}$ be the space of functions

$$DU^p_{\Delta_D} = \{(i\partial_t + \Delta_D)u; u \in U^p_{\Delta_D}\}. \tag{1.15}$$

There is the easy estimate

$$\|u\|_{U^p_\Delta} \lesssim \|u(0)\|_{L^2} + \|(i\partial_t + \Delta_D)u\|_{DU^p_{\Delta_D}}. \tag{1.16}$$

Finally, there is the duality relation

$$(DU^p_{\Delta_D})^* = V^{p'}_{\Delta_D}. \tag{1.17}$$

These spaces are also closed under truncation in time.

$$\begin{aligned} \chi_I &: U^p_{\Delta_D} \to U^p_{\Delta_D}, \\ \chi_I &: V^p_{\Delta_D} \to V^p_{\Delta_D}. \end{aligned} \tag{1.18}$$

Proof: See [20]. □

In particular this implies that if (p, q) is an admissible pair $L^{p'}_t L^{q'}_x \subset V^2_{\Delta_D}$.

Remark: From now on we will understand that U^p_Δ and V^p_Δ refers to $U^p_{\Delta_D}$ and $V^p_{\Delta_D}$ respectively.

We have a Littlewood - Paley type theorem for an exterior domain.

Theorem 1.5 *Let $\Psi \in C_0^\infty$ such that for $\lambda > 0$*

$$\sum_j \Psi(2^{-2j}\lambda) = 1. \tag{1.19}$$

Then for $p \in (1,\infty)$, $f \in C^\infty(\Omega)$

$$\|f\|_{L^p(\Omega)} \sim_p \|(\sum_{j \in \mathbf{Z}} |\Psi(-2^{-2j}\Delta_D)f|^2)^{1/2}\|_{L^p(\Omega)}, \tag{1.20}$$

and for $p \in [2,\infty)$,

$$\|f\|_{L^p(\Omega)} \lesssim_p (\sum_{j \in \mathbf{Z}} \|\Psi(-2^{2j}\Delta_D)f\|_{L^p(\Omega)}^2)^{1/2}. \tag{1.21}$$

Proof: See [22]. □

Remark: As in $\mathbf{R}^d$ let

$$u_N = \Psi(-N^{-2}\Delta_D)u, \tag{1.22}$$

$$u_{\leq 2^j} = \sum_{k=-\infty}^{j} \Psi(-2^{-2k}\Delta_D), \tag{1.23}$$

$$u_{\leq N} + u_{>N} = u. \tag{1.24}$$

It also follows from the fundamental theorem of calculus that

$$\|u_M\|_{L^\infty(\Omega)} \lesssim M^{d/2}\|u_M\|_{L^2(\Omega)}. \tag{1.25}$$

As in the $\mathbf{R}^4$ case, to prove theorem 1.1 it suffices to prove

$$\|u\|_{L^6_{t,x}(\mathbf{R}\times\Omega)} \leq C(E(u_0)) < \infty. \tag{1.26}$$

Let

$$A(E) = \sup\{\|u\|_{L^6_{t,x}(\mathbf{R}\times\Omega)} : u \text{ solves } (1.1), E(u(t)) = E\}. \tag{1.27}$$

For $3 \leq d \leq 6$ it is possible to prove a stability result using exactly the same arguments which are found in [35].

Theorem 1.6 *Suppose that for $3 \leq d \leq 6$ $\tilde{u}$ is an approximate solution to* (1.1) *in that*

$$i\tilde{u}_t + \Delta\tilde{u} = |\tilde{u}|^{\frac{4}{d-2}}\tilde{u} + e, \tag{1.28}$$

$$\tilde{u}|_{\partial\Omega} = 0, \tag{1.29}$$

$$\|\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\Omega)} \leq M, \tag{1.30}$$

$$\|\tilde{u}\|_{L_t^\infty \dot{H}_0^1(I\times\Omega)} \leq E, \tag{1.31}$$

and for some (p,q) admissible

$$\|\nabla e\|_{L_t^{p'} L_x^{q'}(I\times\Omega)} \leq \epsilon, \tag{1.32}$$

for some $\epsilon(M,E) > 0$ sufficiently small. Then there exists a solution $u(t,x)$ to (1.1)*, $u(0,x) = \tilde{u}(0,x)$, such that for (p,q) admissible*

$$\|\nabla[u - \tilde{u}]\|_{L_t^p L_x^q(I\times\Omega)} \leq C(p,E,M)\epsilon. \tag{1.33}$$

Proof: We follow an argument similar to the argument in [35]. Partition I into $\leq C(d,E,M)$ intervals $I_j = [a_j, b_j]$ such that $\|\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I_j\times\Omega)} \leq \delta$ for some small $\delta > 0$. On each I_j

$$\|\nabla\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_j\times\Omega)} \lesssim \|\tilde{u}(a_j)\|_{\dot{H}_0^1(\Omega)} + \|\nabla\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_j\times\Omega)} \|\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I_j\times\Omega)}^{\frac{4}{d-2}} + \epsilon. \tag{1.34}$$

This implies

$$\|\nabla\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I_j\times\Omega)} \lesssim E + \epsilon. \tag{1.35}$$

Partition each I_j into $\leq C(d,E,M)$ subintervals $I_{j,k} = [a_{j,k}, b_{j,k}]$ such that $\|\nabla\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_{j,k}\times\Omega)} \leq \delta$. Now let $v = u - \tilde{u}$. v solves the initial value problem

$$\begin{aligned} (i\partial_t + \Delta)v &= |u|^{\frac{4}{d-2}}u - |\tilde{u}|^{\frac{4}{d-2}}\tilde{u} - e, \\ v(0) &= 0. \end{aligned} \tag{1.36}$$

Using Strichartz estimates,

$$\begin{aligned} \|\nabla v\|_{U_\Delta^2(I_{j,k}\times\Omega)} &\lesssim \|v(a_{j,k})\|_{\dot{H}_0^1(\Omega)} + \\ \|\nabla v\|_{U_\Delta^2(I_{j,k}\times\Omega)} (\|\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I_{j,k}\times\Omega)} &+ \|\nabla\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_{j,k}\times\Omega)} + \|\nabla v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_{j,k}\times\Omega)})^{\frac{4}{d-2}} + \epsilon. \end{aligned} \tag{1.37}$$

This implies that if $\|\nabla v\|_{U^2_\Delta(I_{j,k}\times\Omega)}$ is sufficiently small

$$\|\nabla v\|_{U^2_\Delta(I_{j,k}\times\Omega)} \lesssim \|v(a_{j,k})\|_{\dot{H}^1_0(\Omega)} + \epsilon. \tag{1.38}$$

For $\epsilon(E, M) > 0$ sufficiently small, since $\|v(0)\|_{\dot{H}^1_0(\Omega)} = 0$, and there are finitely many $I_{j,k}$ subintervals,

$$\|\nabla v\|_{U^2_\Delta(I\times\Omega)} \leq C(d, E, M)\epsilon. \tag{1.39}$$

□

Remark: At the present time this stability result cannot be extended to $d > 6$ using the stability arguments of [35] due to the lack of exotic Strichartz estimates in a convex domain.

By theorem 1.6 $A(E)$ is a continuous function of E. This implies $\{E : A(E) = \infty\}$ is a closed set, and therefore has a minimal element E_0. We prove that $E_0 = \infty$.

Suppose $E_0 < \infty$. We use the bilinear virial identities of [29] to prove a bilinear Strichartz estimate for two solutions to the linear problem $iu_t + \Delta u = 0$, $u|_{\partial\Omega} = 0$ outside a convex obstacle. This result combined with theorem 1.6 is enough to prove that a solution u to (1.1) with energy E_0, $\|u\|_{L^6_{t,x}(I\times\Omega)} = M$, M very large, must concentrate at some frequency scale $N(t)$. Partitioning I into subintervals J_k such that $\|u\|_{L^6_{t,x}(I\times\Omega)} = 1$, we see that u must be concentrated at frequency scale $N(t) \sim N_k$ for some N_k. Moreover, some of the solution u must be concentrated at a spatial scale $\sim \frac{1}{N_k}$ for length of time $\sim \frac{1}{N_k^2}$. This combined with the interaction Morawetz estimates of [29] is enough to rule out a quasi - soliton like solution. Conservation of mass rules out a rapid cascade - like solution.

At this point it will be beneficial to say a few words about possible further developments. The purpose of this paper is two - fold. First, it is written to show that the techniques of [18], [17], [19], [16], [40], and [25] require very little in the way of knowledge of the fundamental solution or anything that is extremely Fourier analytic in nature.

The second purpose is to attempt to understand the energy - critical problem in the exterior of a convex obstacle for all $d \geq 3$. The same techniques could yield global well - posedness and scattering for $d = 5$ as well. This will not be discussed in this paper because the fact that $|u|^{\frac{4}{3}}u$ is not an algebraic nonlinearity introduces some additional technical complications. The case $d = 6$ could probably be proved as well, although the proof seems to be hindered by the fact that theorem 1.3 does not include endpoint Strichartz estimates. The case $d = 3$ also seems beyond the reach of the current techniques due to a heavy reliance in [25] on Fourier - analytic techniques to obtain several

key endpoint results. Extending this result to $d > 6$ would likely be far more difficult due to a lack of a stability theorem akin to theorem 1.6.

2 Morawetz Estimates

The mainstay of the argument in this paper is the Morawetz estimates of [29] outside a star shaped obstacle. Therefore, we will summarize the argument before.

Theorem 2.1 *Suppose Σ is a compact star - shaped obstacle and $\Omega = \mathbf{R}^d \setminus \Sigma$ is the exterior to Σ. Let u be a solution to*

$$\begin{aligned} iu_t + \Delta u &= \mu |u|^p u, \\ u|_{t=0} &= u_0. \end{aligned} \tag{2.1}$$

If $d \geq 3$, $\mu \geq 0$,

$$\begin{aligned} &\int_0^T \int_{\partial\Omega} |\partial_n u|^2 dS + \int_0^T \int_\Omega \frac{1}{(1+|x|^2)^{3/2}} (|\nabla u(t,x)|^2 + |u(t,x)|^2) dx dt \\ &+ \frac{\mu p}{p+2} \int_0^T \int_\Omega \frac{1}{(1+|x|^2)^{1/2}} |u(t,x)|^{p+2} dx dt \lesssim \sup_{t \in [0,T]} \|u(t)\|^2_{\dot{H}_0^{1/2}(\Omega)}. \end{aligned} \tag{2.2}$$

Proof: We repeat the proof found in [29]. Let $h(x) = (1+|x|^2)^{1/2}$,

$$M(t) = \int_\Omega h(x) |u(t,x)|^2 dx. \tag{2.3}$$

$$\dot{M(t)} = 2 \int_\Omega h_j(x) Im[\bar{u} \partial_j u](t,x) dx. \tag{2.4}$$

$$\ddot{M(t)} = \int_\Omega h_j(x) [-\partial_k^2 \bar{u} \partial_j u + \bar{u} \partial_j \partial_k^2 u + u \partial_j \partial_k^2 \bar{u} - \partial_k^2 u \partial_j \bar{u}] dx \tag{2.5}$$

$$- \mu \int_\Omega h_j(x) [|u|^p \bar{u} \partial_j u - \bar{u} \partial_j (|u|^p u) - u \partial_j (|u|^p \bar{u}) + |u|^p u \partial_j \bar{u}] dx. \tag{2.6}$$

$$\int_\Omega h_j(x) [\bar{u} \partial_j \partial_k^2 u + u \partial_j \partial_k^2 \bar{u}] dx \tag{2.7}$$

$$= - \int_\Omega h_j(x) [\partial_k \bar{u} \partial_j \partial_k u + \partial_k u \partial_j \partial_k \bar{u}] dx \tag{2.8}$$

$$- \int_\Omega h_{jk}(x) [\bar{u} \partial_j \partial_k u + u \partial_j \partial_k \bar{u}] dx. \tag{2.9}$$

$$(2.9) = 2\int_{\Omega} h_{jk}(x) Re(\partial_k \bar{u} \partial_j u)(t,x) dx - \int_{\Omega} (\Delta\Delta h(x)) |u(t,x)|^2 dx. \tag{2.10}$$

$$(2.8) - \int_{\Omega} h_j(x)[\partial_k^2 \bar{u} \partial_j u + \partial_k^2 u \partial_j \bar{u}](t,x) dx = -2\int_{\Omega} h_j(x) \partial_k Re[(\partial_k \bar{u})(\partial_j u)](t,x) dx \tag{2.11}$$

$$= 2\int_{\Omega} h_{jk}(x) Re[(\partial_k \bar{u})(\partial_j u)](t,x) dx + 2\int_{\partial\Omega} n_k h_j(x) Re[(\partial_k \bar{u})(\partial_j u)](t,x) d\sigma(x), \tag{2.12}$$

where $\vec{n}$ is the outward pointing unit normal to Σ, $d\sigma$ is the surface measure on $\partial\Omega$. Therefore,

$$\begin{aligned}(2.5) = 4\int_{\Omega} h_{jk}(x) Re[(\partial_k \bar{u})(\partial_j u)](t,x) dx - \int_{\Omega} (\Delta\Delta h(x)) |u(t,x)|^2 dx \\ +2\int_{\partial\Omega} n_k h_j(x) Re[(\partial_k \bar{u})(\partial_j u)](t,x) d\sigma(x).\end{aligned} \tag{2.13}$$

$$2\int_{\partial\Omega} n_k h_j(x) Re[(\partial_k \bar{u})(\partial_j u)](t,x) d\sigma(x) = 2\int_{\Omega} h_j(x) Re[(\partial_n \bar{u})(\partial_j u)](t,x) d\sigma(x). \tag{2.14}$$

Because $u|_{\partial\Omega} = 0$, $\nabla u = (\partial_n u)\vec{n}$. Therefore,

$$(2.14) = 2\int_{\partial\Omega} (\partial_n h) |\partial_n u(t,x)|^2 d\sigma(x). \tag{2.15}$$

Let $h(x) = (1+|x|^2)^{1/2}$. $\nabla h(x) = \frac{x}{(1+|x|^2)^{1/2}}$ so $\partial_n h > 0$ on $\partial\Omega$ since Σ is star - shaped.

$$-\Delta\Delta h(x) = (d-1)(d-3)(1+|x|^2)^{-3/2} + 3d(1+|x|^2)^{-7/2} + 3(d-1)(1+|x|^2)^{-7/2} + 6(d-3)|x|^2(1+|x|^2)^{-7/2}. \tag{2.16}$$

(2.16) ≥ 0 for $d \geq 3$.

$$4\int_{\Omega} \partial_k\left(\frac{x_j}{(1+|x|^2)^{1/2}}\right) Re[(\partial_j \bar{u})(\partial_k u)](t,x) dx \tag{2.17}$$

$$= 4\int_{\Omega} \frac{1}{(1+|x|^2)^{1/2}} |\nabla u(t,x)|^2 dx - 4\int_{\Omega} \frac{|x|^2}{(1+|x|^2)^{3/2}} |\partial_r u(t,x)|^2 dx \tag{2.18}$$

$$= 4\int_{\Omega} \frac{1}{(1+|x|^2)^{3/2}} |\partial_r u(t,x)|^2 dx + 4\int_{\Omega} \frac{|x|^2}{(1+|x|^2)^{3/2}} |\nabla\!\!\!/ u(t,x)|^2 dx. \tag{2.19}$$

Finally, integrating by parts,

$$(2.6) = \frac{\mu p}{p+2} \int ((1+|x|^2)^{-3/2} + (d-1)(1+|x|^2)^{-1/2})|u(t,x)|^{p+2} dx. \tag{2.20}$$

Combining (2.13), (2.16), (2.19), and (2.20) proves the theorem. $\Box$

Theorem 2.2 *Suppose Ω is a star - shaped domain. Let $d \geq 1$, u, v be two solutions to*

$$\begin{aligned} iu_t + \Delta u &= \mu |u|^p u, \\ u|_{t=0} &= u_0, \end{aligned} \tag{2.21}$$

$$\begin{aligned} iv_t + \Delta v &= \mu |v|^p v, \\ v|_{t=0} &= v_0. \end{aligned} \tag{2.22}$$

Let $\omega \in S^{d-1}$.

$$4 \int_0^T \int_{\Omega \times \Omega} |\partial_\omega (u(t, x_1 + x^\perp) \bar{v}(t, x_1 + y^\perp))|^2 dx_1 dx^\perp dy^\perp dt \tag{2.23}$$

$$+ \frac{2\mu p}{p+2} \int_0^T \int_{\Omega \times \Omega} |u(t, x_1 + x^\perp)|^2 |v(t, x_1 + y^\perp)|^p dx_1 dx^\perp dy^\perp dt \tag{2.24}$$

$$+ \frac{2\mu p}{p+2} \int_0^T \int_{\Omega \times \Omega} |u(t, x_1 + x^\perp)|^p |v(t, x_1 + y^\perp)|^2 dx_1 dx^\perp dy^\perp dt \tag{2.25}$$

$$\lesssim (\sup_{t \in [0,T]} \|u(t)\|^2_{\dot{H}_0^{1/2}(\Omega)})(\sup_{t \in [0,T]} \|v(t)\|^2_{L^2(\Omega)}) + (\sup_{t \in [0,T]} \|v(t)\|^2_{\dot{H}_0^{1/2}(\Omega)})(\sup_{t \in [0,T]} \|u(t)\|^2_{L^2(\Omega)}). \tag{2.26}$$

Remark: This was also proved in [29].

Proof: Let

$$I_\omega(\mu, u, v) = \int_{\Omega \times \Omega} |(x-y) \cdot \omega| |u(t,x)|^2 |v(t,y)|^2 dx dy. \tag{2.27}$$

Without loss of generality suppose $\omega = (1, 0, ..., 0)$.

$$\dot{I}_\omega(\mu, u, v) = 2 \int_{\Omega \times \Omega} \frac{(x-y)_1}{|(x-y)_1|} Im[\bar{u} \partial_1 u](t,x) |v(t,y)|^2 dx dy + 2 \int \frac{(y-x)_1}{|(y-x)_1|} |u(t,x)|^2 Im[\bar{v} \partial_1 v](t,y) dx dy. \tag{2.28}$$

$$\ddot{I}_\omega(\mu, u, v) = \int_{\Omega \times \Omega} \frac{(x-y)_1}{|(x-y)_1|} |v(t,y)|^2 [-\partial_k^2 \bar{u} \partial_1 u + \bar{u} \partial_1 \partial_k^2 u + u \partial_1 \partial_k^2 \bar{u} - \partial_k^2 u \partial_1 \bar{u}](t,x) dx dy \tag{2.29}$$

$$+\int_{\Omega\times\Omega}\frac{(y-x)_1}{|(y-x)_1|}|u(t,x)|^2[-\partial_k^2\bar{v}\partial_1 v+\bar{v}\partial_1\partial_k^2 v+v\partial_1\partial_k^2\bar{v}-\partial_k^2 v\partial_1\bar{v}](t,y)dxdy \tag{2.30}$$

$$-8\int_{\Omega\times\Omega}Im[\bar{u}\partial_1 u](t,x_1+x^{\perp})Im[\bar{v}\partial_1 v](t,x_1+y^{\perp})dx_1dx^{\perp}dy^{\perp} \tag{2.31}$$

$$-2\mu\int_{\Omega\times\Omega}\frac{(x-y)_1}{|(x-y)_1|}[-|u|^p\bar{u}\partial_1 u+\bar{u}\partial_1(|u|^p u)+u\partial_1(|u|^p\bar{u})-|u|^p u\partial_1\bar{u}](t,x)|v(t,y)|^2dxdy \tag{2.32}$$

$$-2\mu\int_{\Omega\times\Omega}\frac{(y-x)_1}{|(y-x)_1|}|u(t,x)|^2[-|v|^p\bar{u}\partial_1 v+\bar{v}\partial_1(|v|^p v)+v\partial_1(|v|^p\bar{v})-|v|^p v\partial_1\bar{v}](t,y)dxdy. \tag{2.33}$$

Remark: We use the notation $x=x_1+x^{\perp}$, where $x_1=x(x\cdot(1,0,...0))$, $x^{\perp}=x-x_1$, and the notation

$$\int_{\Omega\times\Omega}f(x,y)dx_1dx^{\perp}dy^{\perp}=\int_{-\infty}^{\infty}\int_{x^{\perp}:x_1+x^{\perp}\in\Omega}\int_{y^{\perp}:x_1+y^{\perp}\in\Omega}f(x,y)dy^{\perp}dx^{\perp}dx_1. \tag{2.34}$$

Following the same analysis as in the proof of theorem 2.1,

$$(2.29)=4\int_{\Omega\times\Omega}\partial_k(\frac{(x-y)_1}{|(x-y)_1|})Re[(\partial_k\bar{u})(\partial_1 u)](t,x)|v(t,y)|^2dxdy \tag{2.35}$$

$$-\int_{\Omega\times\Omega}\partial_1(\frac{(x-y)_1}{|(x-y)_1|})\Delta(|u|^2)(t,x)|v(t,y)|^2dxdy \tag{2.36}$$

$$+2\int_{\Omega}|v(t,y)|^2\int_{\partial\Omega}\frac{(x-y)_1}{|(x-y)_1|}Re[(\partial_n\bar{u})(\partial_1 u)](t,x)dxdy. \tag{2.37}$$

Since $\nabla u=(\partial_n u)\vec{n}$, by theorem 2.1

$$2\int_0^T\int_{\Omega}|v(t,y)|^2\int_{\partial\Omega}\frac{(x-y)_1}{|(x-y)_1|}Re[(\partial_n\bar{u})(\partial_1 u)](t,x)d\sigma(x)dy \tag{2.38}$$

$$\lesssim\int_0^T\int_{\Omega}|v(t,y)|^2\int_{\partial\Omega}|\partial_n u(t,x)|^2d\sigma(x)dy\lesssim(\sup_{t\in[0,T]}\|u(t)\|^2_{\dot{H}_0^{1/2}(\Omega)})(\sup_{t\in[0,T]}\|v(t)\|^2_{L^2(\Omega)}). \tag{2.39}$$

This takes care of (2.37). Next,

$$
\begin{aligned}
(2.36) &= -\int_{\Omega\times\Omega} \Delta_x(|u(t,x_1+x^{\perp})|^2)|v(t,x_1+y^{\perp})|^2 dx_1 dx^{\perp} dy^{\perp} \\
&= \int_{\Omega\times\Omega} \partial_{x_1}(|u(t,x_1+x^{\perp})|^2)\partial_{x_1}(|v(t,x_1+y^{\perp})|^2) dx_1 dx^{\perp} dy^{\perp}.
\end{aligned} \tag{2.40}
$$

Finally,

$$
(2.35) = 4\int_{\Omega\times\Omega} |\partial_1 u(t,x+x^{\perp})|^2 |v(t,x_1+y^{\perp})|^2 dx_1 dx^{\perp} dy^{\perp}. \tag{2.41}
$$

Make an identical argument for (2.30).

$$
4|\partial_1 u|^2|v|^2 + 4|\partial_1 v|^2|u|^2 - 8Im[\bar{u}\partial_1 u]Im[\bar{v}\partial v] + 2\partial_1(|u|^2)\partial_1(|v|^2) = 4|\partial_1(u\bar{v})|^2. \tag{2.42}
$$

By the fundamental theorem of calculus,

$$
\sup_{t\in[0,T]} \dot{I}_{\omega}(\mu,u,v) \lesssim (\sup_{t\in[0,T]} \|u(t)\|^2_{\dot{H}_0^{1/2}(\Omega)})(\sup_{t\in[0,T]} \|v(t)\|^2_{L^2(\Omega)}) + (\sup_{t\in[0,T]} \|v(t)\|^2_{\dot{H}_0^{1/2}(\Omega)})(\sup_{t\in[0,T]} \|u(t)\|^2_{L^2(\Omega)}), \tag{2.43}
$$

(2.39), and integrating (2.32) and (2.33) by parts, the proof is complete. □

Corollary 2.3 *When $d = 1$,*

$$
\int_0^T \int_{\Omega} |\partial_x(u\bar{v})|^2(t,x) + \frac{2\mu p}{p+2}|u|^2|v|^{p+2}(t,x) + \frac{2\mu p}{p+2}|u|^{p+2}|v|^2(t,x) dx dt \tag{2.44}
$$

$$
\lesssim (\sup_{t\in[0,T]} \|u(t)\|^2_{\dot{H}_0^{1/2}(\Omega)})(\sup_{t\in[0,T]} \|v(t)\|^2_{L^2(\Omega)}) + (\sup_{t\in[0,T]} \|v(t)\|^2_{\dot{H}_0^{1/2}(\Omega)})(\sup_{t\in[0,T]} \|u(t)\|^2_{L^2(\Omega)}). \tag{2.45}
$$

Now replace $|(x-y)\cdot\omega$ with a more general $\rho(x-y)$ with positive definite Hessian $H_{\rho}(x-y)$.

Theorem 2.4 *Let $F(u,v)(x,y) = \bar{v}(y)\nabla_x u(x) + u(x)\nabla_y \bar{v}(y)$ and $G(u,v)(x,y) = v(y)\nabla_x u(x) - u(x)\nabla_y v(y)$. Then*

$$
4\int_0^T \int_{\Omega\times\Omega} H_{\rho}(x-y)(F(u,v)(x,y), \bar{F}(u,v)(x,y)) dx dy dt \tag{2.46}
$$

$$
+ \frac{\mu p}{p+2}\int_0^T \int_{\Omega\times\Omega} |v|^2(t,y)(\Delta_x \rho)(x-y)|u|^{p+2}(t,x) dx dy dt \tag{2.47}
$$

$$+\frac{\mu p}{p+2}\int_0^T\int_{\Omega\times\Omega}|v|^{p+2}(t,y)(\Delta_x\rho)(x-y)|u|^2(t,x)dxdydt \tag{2.48}$$

$$\lesssim(\sup_{t\in[0,T]}\|u(t)\|_{\dot{H}_0^{1/2}(\Omega)}^2)(\sup_{t\in[0,T]}\|v(t)\|_{L^2(\Omega)}^2)+(\sup_{t\in[0,T]}\|v(t)\|_{\dot{H}_0^{1/2}(\Omega)}^2)(\sup_{t\in[0,T]}\|u(t)\|_{L^2(\Omega)}^2). \tag{2.49}$$

Remark: This also appears in [29].

Proof: Follow the proof of theorem 2.2.

$$\begin{aligned}4\rho_{jk}(x-y)Re((\partial_j\bar{u})(\partial_k u))(t,x)|v(t,y)|^2+4\rho_{jk}(x-y)Re((\partial_j\bar{v})(\partial_k v))(t,y)|u(t,x)|^2\\ -8\rho_{jk}(x-y)Im[\bar{u}\partial_j u](t,x)Im[\bar{v}\partial_k v](t,y)-2\rho_{jk}(x-y)\partial_j(|u|^2)(t,x)\partial_k(|v|^2)(t,y)\\ =4H_\rho(x-y)(F(u,v),\bar{F}(u,v))\\ =4H_\rho(x-y)(G(u,v),\bar{G}(u,v))+\Delta\rho(x-y)\nabla_x(|u|^2)(t,x)\cdot\nabla_y(|v|^2)(t,y).\end{aligned} \tag{2.50}$$

We use these arguments to prove a bilinear Strichartz estimate in the exterior of a star - shaped domain.

Theorem 2.5 *Suppose Ω is a domain exterior to a star shaped obstacle. Suppose $u_0=\Psi(-M^{-2}\Delta_D)u_0$ and $v_0=\Psi(-N^{-2}\Delta_D)v_0$, $M\leq N$. If u and v are linear solutions to*

$$\begin{aligned}iu_t+\Delta u=0,\\ u(0)=u_0,\end{aligned} \tag{2.51}$$

$$\begin{aligned}iv_t+\Delta v=0,\\ v(0)=v_0,\end{aligned} \tag{2.52}$$

then

$$\|\nabla(u\bar{v})\|_{L^2_{t,x}(\mathbf{R}\times\Omega)}\lesssim M^{(d-1)/2}N^{1/2}\|u_0\|_{L^2(\Omega)}\|v_0\|_{L^2(\Omega)}. \tag{2.53}$$

Proof: By elementary Strichartz estimates the theorem follows for $M\sim N$. By the fundamental theorem of calculus, when $d=1$, $\|u\|^2_{L^\infty(\mathbf{R})}\lesssim\|\nabla u\|_{L^2(\mathbf{R})}\|u\|_{L^2(\mathbf{R})}$. In general,

$$\|u\|^2_{L^\infty(\Omega)}\lesssim\|(-\Delta_D)^{d/2}u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}\lesssim\frac{1}{M^d}\|(-\Delta_D)^{d/2}u\|^2_{L^2(\Omega)}+M^d\|u\|^2_{L^2(\Omega)}. \tag{2.54}$$

Let $\chi\in C_0^\infty(\mathbf{R}^d)$, $\chi=1$ on $|x|\leq\frac{1}{2}$, $\chi=0$ on $|x|>1$. For $x_0\in\Omega$,

$$|u(x_0)|^2 \lesssim \frac{1}{M^d}\|\chi(\frac{M(x-x_0)}{2})((-\Delta_D)^{d/2}u)\|_{L^2(\Omega)}^2 + M^d\|\chi(\frac{M(x-x_0)}{2})u\|_{L^2(\Omega)}^2. \tag{2.55}$$

Making basic Strichartz estimates,

$$M^d \int_0^T \int_\Omega |u(t,x_1+x^\perp)|^2 \int_{|\tau|\leq \frac{2}{M}} \int_{|y^\perp - x^\perp|\leq \frac{2}{M}} |\partial_1 v(t,x_1+\tau e_1+y^\perp)|^2 dy^\perp d\tau dx^\perp dx_1 \tag{2.56}$$

$$\lesssim M^{d-1}\|u_0\|_{L^2(\Omega)}^2 \|v_0\|_{L^2(\Omega)}^2. \tag{2.57}$$

Therefore, by theorem 2.2, (2.57), combined with the fact that $|(x-y)_1 - \tau|$ has a positive definite Hessian,

$$M^d \int_0^T \int_\Omega |\partial_1(u(t,x_1+x^\perp))|^2 \int_{|\tau|\leq \frac{2}{M}} \int_{|y^\perp - x^\perp|\leq \frac{2}{M}} |v(t,x_1+\tau e_1+y^\perp)|^2 dy^\perp d\tau dx^\perp dx_1 dt \tag{2.58}$$

$$\begin{aligned} &\lesssim M^{d-1}(\sup_{t\in[0,T]}\|u(t)\|_{L^2(\Omega)}^2)(\sup_{t\in[0,T]}\|v(t)\|_{\dot{H}_0^{1/2}(\Omega)}^2) \\ &+M^{d-1}(\sup_{t\in[0,T]}\|v(t)\|_{L^2(\Omega)}^2)(\sup_{t\in[0,T]}\|u(t)\|_{\dot{H}_0^{1/2}(\Omega)}^2) \lesssim M^{d-1}N\|u_0\|_{L^2(\Omega)}^2\|v_0\|_{L^2(\Omega)}^2. \end{aligned} \tag{2.59}$$

Similarly, since Δ_D commutes with the solution operator to (2.52),

$$\frac{1}{M^d} \int_0^T \int_\Omega |\partial_1(u(t,x_1+x^\perp))|^2 \int_{|\tau|\leq \frac{2}{M}} \int_{|y^\perp - x^\perp|\leq \frac{2}{M}} |(-\Delta_D)^{d/2} v(t,x_1+\tau e_1+y^\perp)|^2 dy^\perp d\tau dx^\perp dx_1 dt \tag{2.60}$$

$$\begin{aligned} &\lesssim \frac{1}{M^{d+1}}(\sup_{t\in[0,T]}\|u(t)\|_{L^2(\Omega)}^2)(\sup_{t\in[0,T]}\|(-\Delta_D)^{d/2}v(t)\|_{\dot{H}_0^{1/2}(\Omega)}^2) \\ &+\frac{1}{M^{d+1}}(\sup_{t\in[0,T]}\|(-\Delta_D)^{d/2}v(t)\|_{L^2(\Omega)}^2)(\sup_{t\in[0,T]}\|u(t)\|_{\dot{H}_0^{1/2}(\Omega)}^2) \lesssim M^{d-1}N\|u_0\|_{L^2(\Omega)}^2\|v_0\|_{L^2(\Omega)}^2. \end{aligned} \tag{2.61}$$

This completes the proof of theorem 2.5. □

We can now prove that a solution to (1.1) with energy E_0 and very high $\|u\|_{L_{t,x}^6(J\times\Omega)}$ norm for some compact interval J must concentrate in frequency.

By theorem 2.5 and theorem 1.6,

Theorem 2.6 *Suppose u solves* (1.1), $E(u(t)) = E_0$,

$$\|u\|_{L^6_{t,x}(J \times \Omega)} = M, \tag{2.62}$$

for some M very large. Then fix $\delta(E_0) > 0$ sufficiently small and partition J into subintervals J_k such that

$$\|u\|_{L^6_{t,x}(J_k \times \Omega)} = \delta. \tag{2.63}$$

For each J_k there exists $N_k \in (0, \infty)$ such that for all $\eta > \eta_0(M)$, $\eta_0(M) \searrow 0$ as $M \nearrow \infty$, there exists $C(\eta) < \infty$ such that

$$\|P_{\leq \frac{1}{C(\eta)} N_k} u\|_{L^\infty_t \dot{H}^1(\Omega)(J_k \times \Omega)} + \|P_{\leq \frac{1}{C(\eta)} N_k} u\|_{L^\infty_t L^4_x(J_k \times \Omega)} < \eta. \tag{2.64}$$

$$\|P_{\geq C(\eta) N_k} u\|_{L^\infty_t \dot{H}^1(J_k \times \Omega)} + \|P_{\geq C(\eta) N_k} u\|_{L^\infty_t L^4_x(J_k \times \Omega)} < \eta. \tag{2.65}$$

Moreover, if J_k and J_{k+1} are adjacent intervals then $N_k \sim_{E_0} N_{k+1}$.

We divide into two cases. Let $C_0 = \inf_{t \in J} N(t)$, M very large. We treat the case

$$C_0^3 \sum_{J_k \subset J} \frac{1}{N_k^3} \leq K_0 \tag{2.66}$$

as [40], [25] treated the rapid cascade. We treat

$$C_0^3 \sum_{J_k \subset J} \frac{1}{N_k^3} \geq K_0 \tag{2.67}$$

as [40], [25] treated the pseudo - soliton, for some K_0 to be specified later. We prove,

Theorem 2.7 *There does not exist a solution to* (1.1),

$$\|u\|_{L^6_{t,x}(J \times \Omega)} = M, \tag{2.68}$$

$E(u(t)) = E_0$, *for M very large.*

3 Long time Strichartz estimates for the rapid cascade

We first rule out a scenario similar to the rapid frequency cascade. Fix K.

Theorem 3.1 *There is a constant $K_0(M)$ such that $K_0(M) \nearrow \infty$ as $M \to \infty$ and there does not exist a solution to* (1.1) *with energy $E(u(t)) = E_0$, an interval J with*

$$C_0^3 \sum \frac{1}{N_k^3} \leq K_0(M), \tag{3.1}$$

$$\|u\|_{L^6_{t,x}(J \times \Omega)} = M. \tag{3.2}$$

To prove this theorem we utilize a slight modification of the arguments of [40], [25] for the energy - critical problem in $\mathbf{R}^d$. See also [18], [17], and [16] for induction on frequency in the mass - critical case.

As in the case of the energy - critical nonlinear Schrödinger equation on flat space we will rule out a sufficiently large blowup solution.

Theorem 3.2 *Suppose J is a union of subintervals J_k such that for some $\epsilon > 0$*

$$\sum \frac{1}{N_k^{2(2-\epsilon)}} = K. \tag{3.3}$$

Then if (p, q) satisfies (1.9),

$$\|\nabla P_{\leq N} u\|_{L_t^p L_x^q(J \times \Omega)} \lesssim_{\epsilon,p,q,d} 1 + K^{1/p} N^{\frac{2(2-\epsilon)}{p}}. \tag{3.4}$$

Proof: Fix $\epsilon > 0$. For each dyadic N we partition J at level N. We call these corresponding intervals J_N^l. If $N_k > \frac{c}{N}$, for some small, fixed $c > 0$ to be specified later, then we say J_k is a bad interval, $J_{N,b}^l$. We group the remaining J_k subintervals into good $J_{N,g}^l$ intervals such that each good interval satisfies

$$\frac{1}{2} < N^{2(2-\epsilon)} \sum_{J_k \subset J_{N,g}^l} \frac{1}{N_k^{2(2-\epsilon)}} \leq 1, \tag{3.5}$$

or

$$N^{2(2-\epsilon)} \sum_{J_k \subset J_{N,g}^l} \frac{1}{N_k^{2(2-\epsilon)}} \leq \frac{1}{2}, \tag{3.6}$$

and $J_{N,g}^l$ is adjacent to a bad interval. It suffices to prove

Lemma 3.3 *For any dyadic integer N and any interval J_N^l,*

$$\|\nabla P_{\leq N} u\|_{U^2_\Delta(J_N^l \times \Omega)} \lesssim_{E_0,\epsilon} 1. \tag{3.7}$$

Indeed, for any $p > 2$,

$$\|\nabla P_{\leq N} u\|_{L_t^p L_x^q(J_N^l \times \Omega)} \lesssim \|\nabla P_{\leq N} u\|_{U_\Delta^2(J_N^l \times \Omega)} \lesssim_{E_0, \epsilon} 1. \tag{3.8}$$

Since $\sharp\{J_N^l\} \lesssim N^{2(2-\epsilon)} K$, summing up the norm of J_N^l intervals in l^p proves theorem 3.2. $\Box$

Proof of lemma 3.3*:* This follows from Duhamel's formula. For $t_{0,N}^l \in J_N^l$, a solution to (1.1) satisfies

$$e^{i(t - t_{0,N}^l)\Delta} u(t_{0,N}^l) + \int_{t_{0,N}^l}^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau, \tag{3.9}$$

where $F(u) = |u|^2 u$. By conservation of energy, $\|\nabla e^{i(t - t_{0,N}^l)\Delta} u(t_{0,N}^l)\|_{U_\Delta^2(J_N^l \times \Omega)} \lesssim 1$.

Next, for a bad interval $J_{N,b}^l$,

$$\|\nabla F(u)\|_{L_t^{p'} L_x^{q'}(J_{N,b}^l \times \Omega)} \lesssim \|\nabla F(u)\|_{L_{t,x}^3(J_{N,b}^l \times \Omega)} \|u\|_{L_{t,x}^6(J_{N,b}^l \times \Omega)}^2 \lesssim 1. \tag{3.10}$$

For $N = N_{max} = \sup_k N_k$ all intervals are bad, and so we are done. We therefore proceed by induction. Let C be some large, fixed constant, $c(C) > 0$ a small constant to be chosen momentarily.

$$\|\nabla F(P_{<CN} u)\|_{L_t^{2-} L_x^{4/3+}(J_{N,g}^l \times \Omega)} \lesssim \tag{3.11}$$

$$\|P_{\leq CN} u\|_{L_t^{2+} L_x^{4-}(J_{N,g}^l \times \Omega)}^{1+} \|P_{<cN_k} u\|_{L_t^\infty L_x^4(J_{N,g}^l \times \Omega)}^{2-} \lesssim (C)^{1+} \eta^{2-}. \tag{3.12}$$

Next let $p = 2+$, (p, q) satisfies (1.9), $p \searrow 2$ as $\epsilon \searrow 0$. By induction, for $M > CN$, interpolating U_Δ^2 estimates with $L_t^\infty \dot{H}^1$ estimates,

$$\|u_M\|_{L_t^p L_x^q(J_{N,g}^l \times \Omega)}^{1+} \|u_M\|_{L_t^\infty L_x^2(J_{N,g}^l \times \Omega)}^{1-} \|u_M\|_{L_t^\infty L_x^4(J_{N,g}^l \times \Omega)} \lesssim \frac{1}{M^2} (\frac{M}{N})^{(2-\epsilon)(1+)} (\eta(c))^{2-}. \tag{3.13}$$

Therefore, by Sobolev embedding,

$$N \|P_{<N} F(u_{CN < \cdot < cN_k})\|_{L_t^{p'} L_x^{q'}(J_{N,g}^l \times \Omega)} \lesssim (\eta(c))^{2-}. \tag{3.14}$$

Choosing η sufficiently small, $c(\eta) > 0$ closes the induction. Next, for some $\delta(p, \epsilon) > 0$,

$$\begin{aligned} N^2 \|u_{>cN_k}\|_{L_t^p L_x^q(J_{N,g}^l \times \Omega)}^{1+\delta} \|u_{>cN_k}\|_{L_t^\infty L_x^2(J_{N,g}^l \times \Omega)}^{1-\delta} \|u_{>cN_k}\|_{L_t^\infty L_x^4(J_{N,g}^l \times \Omega)} \\ \lesssim N^2 (\sum \frac{1}{(cN_k)^{2(2-\frac{\epsilon}{2})}})^{\frac{2}{2-\frac{\epsilon}{2}}} \lesssim \frac{1}{c^2}. \end{aligned} \tag{3.15}$$

Notice that this term does not depend on the inductive hypotheses. The last inequality follows from the fact that $\frac{N}{N_k} \leq 1$ on good intervals. Finally, by the inductive hypothesis

$$N \| |u_{>CN}| |u_{<CN}|^2 + |u_{>CN}|^2 |u_{<CN}| \|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}(J_{N,g}^l)} \tag{3.16}$$

$$\begin{aligned} &\lesssim N \|u_{<CN}\|_{L_t^{2+} L_x^{\infty-}(J_{N,g}^l \times \Omega)}^{1+} \|u_{<CN}\|_{L_t^\infty L_x^4(J_{N,g}^l \times \Omega)}^{1-} \|u_{>CN}\|_{L_t^\infty L_x^2(J_{N,g}^l \times \Omega)} \\ &+ N^2 \|u_{>CN}\|_{L_t^{2+} L_x^{4-}(J_{N,g}^l \times \Omega)}^{1+} \|u_{>CN}\|_{L_t^\infty L_x^2(J_{N,g}^l \times \Omega)}^{1-} \|u_{<CN}\|_{L_t^\infty L_x^4(J_{N,g}^l \times \Omega)} \lesssim C^{1+}\eta^{1-} + \eta \frac{1}{C^2}. \end{aligned} \tag{3.17}$$

Therefore we have proved

$$\|\nabla P_{<N}[F(u) - F(u_{<CN})]\|_{L_t^{p'} L_x^{q'}(J_{N,g}^l \times \Omega)} \lesssim 1. \tag{3.18}$$

This completes the proof of lemma 3.3. $\Box$

Remark: As in [18] we can upgrade (3.4) to

$$\|\nabla P_{\leq N} u\|_{L_t^p L_x^q(J \times \Omega)} \lesssim_{\epsilon,p,q,d} \sigma_J(N) + K^{1/p} N^{\frac{2(1+\frac{4}{d-2}-\epsilon)}{p}}, \tag{3.19}$$

where $\sigma_J(N)$ is a frequency envelope that majorizes

$$\inf_{t \in J} \|\nabla P_{\leq N} u(t)\|_{L_t^\infty L_x^2(J \times \Omega)}. \tag{3.20}$$

Let

$$\sigma_J(N) = \sum_{j=-\infty}^{\infty} 2^{-|j|} \inf_{t \in J} \|P_{2^j N} u(t)\|_{L_t^\infty \dot{H}_x^1(J \times \Omega)}. \tag{3.21}$$

This is enough to prove theorem 3.1.

Proof of theorem 3.1*:* Let u be a solution to (1.1), $\|u\|_{L_{t,x}^6(J \times \Omega)} = M$, $K = \frac{K_0}{C_0^3}$. For any N let $u_l = P_{\leq N} u$, $u_l + u_h = u$. If $KN^3 \leq 1$, then as in the previous section

$$\frac{d}{dt}\langle \nabla P_{\leq N} u, \nabla P_{\leq N} u\rangle = 2i \langle \nabla P_{\leq N} F(u), \nabla P_{\leq N} u\rangle \lesssim \sigma_J(N)^2 + KN^3. \tag{3.22}$$

This implies that for any $t \in J$,

$$u(t) = v(t) + w(t), \tag{3.23}$$

where

$$\|v(t)\|_{L_t^\infty \dot{H}_x^{-1/4}(J\times\Omega)} \lesssim \frac{K_0^{5/12}}{C_0^{5/4}}, \tag{3.24}$$

and $\|w(t)\|_{L_t^\infty \dot{H}_x^1(J\times\Omega)} \searrow 0$ as $M \nearrow \infty$. However, by interpolation this implies that for $t \in J_k$, $\eta > \eta_0(M)$,

$$\|v(t)\|_{L_x^2(\Omega)} \lesssim \frac{C(\eta)}{N_k} + \eta^{1/5}\frac{K_0^{1/3}}{C_0}. \tag{3.25}$$

Since J satisfies (3.1), this implies that for M sufficiently large there exists an interval J_k such that for $t \in J_k$, for any $\eta > \eta(M) > 0$,

$$\|v(t)\|_{L_x^2(\Omega)} << \frac{1}{C_0}. \tag{3.26}$$

Moreover, for some $\tilde{t}$ in this interval, with t_0 satisfying $N(t_0) = \inf_{t\in J} N(t)$, and without loss of generality $t_0 < \tilde{t}$,

$$\|u\|_{L_{t,x}^6([t_0,\tilde{t}]\times\Omega)} \lesssim_{K_0} 1. \tag{3.27}$$

By the perturbation lemma this contradicts theorem 2.6. $\Box$

4 Interaction Morawetz estimates

Theorem 4.1 *There is a fixed constant $K_0 < \infty$ such that for M sufficiently large, there does not exist a solution to* (1.1) *satisfying $E(u(t)) = E_0$,*

$$C_0^3 \sum_{J_k \subset J} \frac{1}{N_k^3} \geq K_0, \tag{4.1}$$

and

$$\|u\|_{L_{t,x}^6(J\times\Omega)} = M. \tag{4.2}$$

Recall that $K = \frac{K_0}{C_0^3}$.

Theorem 4.2 *Let $u_h = P_{\geq K^{-1/3}} u$, $u_h + u_l = u$.*

$$\int_J \int_{\partial\Omega} |\partial_n u_h(t,x)|^2 dS_x dt \lesssim K^{1/3}. \tag{4.3}$$

Proof: By §3, for (p,q) satisfying (1.9), $p>2$,

$$\|\nabla u_l\|_{L_t^p L_x^q(J\times\Omega)} \lesssim 1. \tag{4.4}$$

We will also postpone the proof of the estimate

$$\|u_h\|_{L^3_{t,x}(J\times\Omega)} + \|u_h\|_{L_t^\infty L_x^2(J\times\Omega)} \lesssim K^{1/3}. \tag{4.5}$$

Let

$$M(t) = \int_\Omega \frac{x_j}{(1+|x|^2)^{1/2}} Im[\bar{u}_h \partial_j u_h](t,x)dx. \tag{4.6}$$

$$\sup_{t\in J} |M(t)| \lesssim K^{1/3}. \tag{4.7}$$

Therefore, following the analysis in [29],

$$\int_J \int_{\partial\Omega} |\partial_n u_h(t,x)|^2 dS_x dt + \int_J \int \frac{1}{(1+|x|^2)^{3/2}} (|\nabla u_h(t,x)|^2 + |u_h(t,x)|^2) dxdt \tag{4.8}$$

$$+ \int_J \int_\Omega \frac{x_j}{(1+|x|^2)^{1/2}} \{P_h F(u), u_h\}_j dxdt \lesssim K^{1/3}, \tag{4.9}$$

where

$$\{u,v\}_j = Re[\bar{u}\partial_j v - \bar{v}\partial_j u]. \tag{4.10}$$

$$\{P_h F(u), u_h\}_j = \{F(u), u\}_j - \{F(u_l), u_l\}_j - \{F(u) - F(u_l), u_l\}_j - \{P_l F(u), u_h\}_j. \tag{4.11}$$

$$\{F(u), u\}_j - \{F(u_l), u_l\}_j = \frac{1}{2}\partial_j [|u(t,x)|^4 - |u_l(t,x)|^4]. \tag{4.12}$$

$$\int_J \int_\Omega \frac{x_j}{(1+|x|^2)^{1/2}} [F(u) - F(u_l)](\partial_j u_l)(t,x) dxdt \tag{4.13}$$

$$\lesssim \|\nabla u_l\|_{L_t^3 L_x^{12}} \|u_h\|_{L^3_{t,x}}^2 \|u_h\|_{L_t^\infty L_x^4} + \|\nabla u_l\|_{L^3_{t,x}} \|u_l\|_{L_t^3 L_x^{12}}^2 \|u_h\|_{L_t^\infty L_x^2} \lesssim K^{1/3}. \tag{4.14}$$

Next, by Sobolev embedding,

$$\int_J \int_\Omega \frac{x_j}{(1+|x|^2)^{1/2}} (u_h)(\partial_j P_l F(u))(t,x) dxdt \lesssim \|u_h\|_{L_t^\infty L_x^2} \|\nabla u_l\|_{L^3_{t,x}} \|u_l\|_{L_t^3 L_x^{12}}^2 \tag{4.15}$$

$$+ K^{-1/3}\|u_h\|_{L^3_{t,x}}^2 \|u_l\|_{L_t^3 L_x^{12}} \|u_l\|_{L_t^\infty L_x^4} + K^{-2/3}\|u_h\|_{L^3_{t,x}}^3 \|u_h\|_{L_t^\infty L_x^4} \lesssim K^{1/3}. \tag{4.16}$$

Now consider

$$\int_J \int_\Omega \frac{x_j}{(1+|x|^2)^{1/2}}(u_l)\partial_j(F(u)-F(u_l))(t,x)dxdt + \int_J \int_\Omega \frac{x_j}{(1+|x|^2)^{1/2}} P_l F(u)\partial_j u_h(t,x)dxdt. \tag{4.17}$$

Integrating by parts, since we have already considered (4.13), (4.15), it only remains to consider when ∂_j hits $\frac{x_j}{(1+|x|^2)^{1/2}}$. Therefore we have proved

$$\int_J \int_{\partial\Omega} |\partial_n u(t,x)|^2 dS_x dt + \int_J \int_\Omega \frac{1}{(1+|x|^2)^{3/2}}[|u_h(t,x)|^2 + |\nabla u_h(t,x)|^2]dxdt \tag{4.18}$$

$$\int_J \int_\Omega \frac{1}{(1+|x|^2)^{1/2}}|u_h(t,x)|^4 dxdt + \int_J \int_\Omega \frac{1}{(1+|x|^2)^{1/2}}(u_h)P_l F(u_h)(t,x)dxdt \tag{4.19}$$

$$+ \int_J \int_\Omega \frac{1}{(1+|x|^2)^{1/2}} O(|u_h(t,x)||u_l(t,x)|^3) + O(|u_l(t,x)||u_h(t,x)|^3)dxdt \lesssim K^{1/3}. \tag{4.20}$$

By Hardy's inequality, that is for $0 \leq s < \frac{d}{2}$,

$$\|\frac{1}{|x|^s} f\|_{L^2(\mathbf{R}^d)} \lesssim_{s,d} \||\nabla|^s f\|_{L^2(\mathbf{R}^d)}, \tag{4.21}$$

and Sobolev embedding,

$$\int_J \int_\Omega \frac{1}{(1+|x|^2)^{1/2}}(u_h)P_l F(u_h)(t,x)dxdt \lesssim K^{2/3}\|\frac{1}{|x|}u_h\|_{L_t^\infty L_x^2}\|u_h\|_{L^3_{t,x}}^3 \lesssim K^{1/3}. \tag{4.22}$$

Finally,

$$(4.20) \leq \epsilon \int_J \int_\Omega \frac{1}{(1+|x|^2)^{1/2}}|u_h(t,x)|^4 dxdt + C(\epsilon)\int_J \int_\Omega \frac{1}{(1+|x|^2)^{1/2}}|u_h(t,x)||u_l(t,x)|^3 dxdt. \tag{4.23}$$

$$\int_J \int_\Omega \frac{1}{(1+|x|^2)^{1/2}}|u_h(t,x)||u_l(t,x)|^3 dxdt \lesssim \|u_h(t,x)\|_{L_t^\infty L_x^2}\|\frac{1}{|x|}|u_l|^3\|_{L_t^1 L_x^2} \tag{4.24}$$

$$\lesssim \|u_h(t,x)\|_{L_t^\infty L_x^2}\|\nabla u_l\|_{L^3_{t,x}}\|u_l\|_{L_t^3 L_x^{12}}^2 \lesssim K^{1/3}. \tag{4.25}$$

Choosing $\epsilon > 0$ sufficiently small and fixed completes the proof of theorem 4.2. $\Box$

Theorem 4.3 *For $\|u\|_{L^6_{t,x}(J\times\Omega)} = M$, M very large, u solves* (1.1),

$$\||\nabla|^{-1/2}|u_h(t,x)|^2\|^2_{L^2_{t,x}(J\times\Omega)} \lesssim \frac{o(K_0)}{C_0^3}. \tag{4.26}$$

$\frac{o(K_0)}{K_0} \to 0$ as $K_0 \nearrow \infty$.

Proof: We build on the arguments of [29]. take the interaction Morawetz quantity

$$M(t) = \int_{\Omega\times\Omega} \frac{(x-y)_j}{|x-y|} Im[\bar{u}_h(t,x)\partial_j u_h(t,x)]|u_h(t,y)|^2 dxdy. \tag{4.27}$$

$$\dot{M}(t) = \int_{\Omega\times\Omega} \frac{(x-y)_j}{|x-y|} Re[-\partial_k^2 \bar{u}_h(t,x)\partial_j u_h(t,x) + \bar{u}_h(t,x)\partial_j\partial_k^2 u_h(t,x)]|u_h(t,y)|^2 dxdy \tag{4.28}$$

$$+ \int_{\Omega\times\Omega} \frac{(x-y)_j}{|x-y|} \{P_h(|u|^2 u), u_h\}_j(t,x)|u_h(t,y)|^2 dxdy \tag{4.29}$$

$$- 2\int_{\Omega\times\Omega} \frac{(x-y)_j}{|x-y|} Im[\bar{u}_h(t,x)\partial_j u_h(t,x)]\partial_k Im[\bar{u}_h(t,y)\partial_k u_h(t,y)] dxdy \tag{4.30}$$

$$+ 2\int_{\Omega\times\Omega} \frac{(x-y)_j}{|x-y|} Im[\bar{u}_h(t,x)\partial_j u_h(t,x)] Im[P_h(|u|^2 u)u_h](t,y) dxdy. \tag{4.31}$$

Integrating by parts, since $u|_{\partial\Omega} = 0$,

$$(4.28) = \int_{\Omega\times\Omega} \partial_k(\frac{(x-y)_j}{|x-y|}) Re[\partial_k \bar{u}_h(t,x)\partial_j u_h(t,x)]|u_h(t,y)|^2 dxdy \tag{4.32}$$

$$+ \int_\Omega |u_h(t,y)|^2 \int_{\partial\Omega} \frac{(x-y)_j}{|x-y|} \nu_k Re[\partial_k \bar{u}_h(t,x)\partial_j(t,x)] dS_x dy \tag{4.33}$$

$$- \int_{\Omega\times\Omega} \partial_k(\frac{(x-y)_j}{|x-y|}) Re[\bar{u}_h(t,x)\partial_j\partial_k u_h(t,x)]|u_h(t,y)|^2 dxdy, \tag{4.34}$$

where ν_k is the outward pointing unit normal to $\partial\Omega$. By theorem 2.6 $\|u_h\|_{L_t^\infty L_x^2} \lesssim \frac{o(K_0^{1/3})}{C_0}$, so by theorem 4.2,

$$\int_J \int_\Omega |u_h(t,y)|^2 \int_{\partial\Omega} \frac{(x-y)_j}{|x-y|} \nu_k Re[\partial_k \bar{u}_h(t,x)\partial_j u_h(t,x)] dS_x dy \lesssim \frac{o(K_0)}{C_0^3}. \tag{4.35}$$

Next, integrating by parts,

$$(4.34) = \int_{\Omega\times\Omega} \partial_k(\frac{(x-y)_j}{|x-y|}) Re[\partial_j \bar{u}_h(t,x)\partial_k u_h(t,x)]|u_h(t,y)|^2 dxdy \tag{4.36}$$

$$-\frac{1}{2}\int_{\Omega\times\Omega}(\Delta\Delta|x-y|)|u_h(t,x)|^2|u_h(t,y)|^2 dxdy. \tag{4.37}$$

As in $\mathbf{R}^d$,

$$\begin{aligned}&\int_{\Omega\times\Omega}\partial_k(\frac{(x-y)_j}{|x-y|})(Re[\partial_j\bar{u}_h(t,x)\partial_k u_h(t,x)]|u_h(t,y)|^2 \\ &-Im[\bar{u}_h(t,x)\partial_j u_h(t,x)]Im[\bar{u}_h(t,y)\partial_k u_h(t,y)])dxdy \geq 0.\end{aligned} \tag{4.38}$$

Therefore, combining the analysis in theorem 4.2 with $\|u_h\|_{L_t^\infty L_x^2} \lesssim \frac{o(K_0^{1/3})}{C_0}$,

$$\int_J\int_{\Omega\times\Omega}(-\Delta\Delta|x-y|)|u_h(t,x)|^2|u_h(t,y)|^2 dxdydt + \int_J\int_{\Omega\times\Omega}\frac{1}{|x-y|}|u_h(t,x)|^4|u_h(t,y)|^2 dxdydt \tag{4.39}$$

$$+\int_J\int_{\Omega\times\Omega}\frac{(x-y)_j}{|x-y|}Im[\bar{u}_h(t,x)\partial_j u_h(t,x)]Im[P_h(|u|^2u)\bar{u}_h](t,y)dxdydt \lesssim \frac{o(K_0)}{C_0^3}. \tag{4.40}$$

$$Im[P_h(|u|^2u)\bar{u}_h] = Im[|u|^4-|u_l|^4-(F(u)-F(u_l))\bar{u}_l-P_l(|u|^2u)\bar{u}_h] = -Im[(F(u)-F(u_l))\bar{u}_l+P_lF(u)\bar{u}_h]. \tag{4.41}$$

By Sobolev embedding

$$\|u_h^3u_l+P_l(u_h^3)u_h\|_{L^1_{t,x}(J\times\Omega)} \lesssim \|u_h\|^3_{L^3_{t,x}}[\|u_l\|_{L^\infty_{t,x}}+K^{-1/3}\|u_h\|_{L_t^\infty L_x^4}] \lesssim K^{2/3}. \tag{4.42}$$

$$\|u_h^2u_l^2+P_l(u_h^2u_l)u_h\|_{L^1_{t,x}(J\times\Omega)} \lesssim \|u_h\|^2_{L^3_{t,x}}[\|u_l\|^2_{L^6_{t,x}}+\|u_h\|_{L^3_{t,x}}\|u_l\|_{L^\infty_{t,x}}] \lesssim K^{2/3}. \tag{4.43}$$

$$\|P_l(u_hu_l^2)u_h\|_{L^1_{t,x}(J\times\Omega)} \lesssim \|u_h\|^2_{L^3_{t,x}}\|u_l\|^2_{L^6_{t,x}} \lesssim K^{2/3}. \tag{4.44}$$

Finally, for $u_h = \frac{\Delta_D}{\Delta_D}u_h$, integrating by parts,

$$\int_J\int_{\Omega\times\Omega}\frac{(x-y)_j}{|x-y|}Im[\bar{u}_h(t,x)\partial_j u_h(t,x)][u_h(t,y)u_l(t,y)^3+P_lF(u_l)(t,y)u_h(t,y)] \tag{4.45}$$

$$= \int_J \int_{\Omega\times\Omega} \frac{(x-y)_j}{|x-y|} Im[\bar{u}_h(t,x)\partial_j u_h(t,x)](\frac{\partial_k}{\Delta_D} u_h(t,y))\partial_k(u_l^3 + P_l F(u_l))(t,y)dxdy \tag{4.46}$$

$$+ \int_J \int_{\Omega\times\Omega} \frac{1}{|x-y|} Im[\bar{u}_h(t,x)\nabla u_h(t,x)](\frac{\nabla}{\Delta_D} u_h(t,y))(u_l^3 + P_l F(u_l))(t,y)dxdy. \tag{4.47}$$

$$(4.46) \lesssim K^{1/3} \|u_h\|_{L_t^\infty L_x^2}^2 \|\nabla u_h\|_{L_t^\infty L_x^2} \|\nabla u_l\|_{L_{t,x}^3} \|u_l\|_{L_t^3 L_x^{12}}^2 \lesssim \frac{o(K_0)}{C_0^3}. \tag{4.48}$$

Now by the Hardy - Littlewood - Sobolev inequality,

$$(4.47) \lesssim K^{1/3} \|\nabla u_h\|_{L_t^\infty L_x^2} \|u_h\|_{L_{t,x}^3}^2 \|u_l\|_{L_t^3 L_x^{12}} \|u_l\|_{L_t^\infty L_x^4}^2 \lesssim \frac{o(K_0)}{C_0^3}. \tag{4.49}$$

This proves theorem 4.3. $\Box$

Proof of theorem 4.1*:* Now we need some constants

$$0 < \eta_1 << \eta << 1. \tag{4.50}$$

Because

$$K_0 = C_0^3 \sum_{J_k \subset J} \frac{1}{N_k^3}, \tag{4.51}$$

for $\eta_1 > 0$ there exists $K_0(\eta_1)$ sufficiently large such that there exists $J_k \subset J$ with

$$\| |\nabla|^{-1/2} |u_h|^2 \|_{L_{t,x}^2(J_k\times\Omega)}^2 \leq \frac{\eta_1}{N_k^3}. \tag{4.52}$$

Now on each J_k,

$$\|\nabla |u_h|^2\|_{L_{t,x}^2(J_k\times\Omega)} \lesssim \|\nabla u_h\|_{L_{t,x}^3} \|u_h\|_{L_{t,x}^6} \lesssim 1. \tag{4.53}$$

Therefore, by interpolation

$$\|u_h\|_{L_{t,x}^4(J_k\times\Omega)}^4 \lesssim \frac{\eta_1^{2/3}}{N_k^2}. \tag{4.54}$$

Now by interpolation, since $\|u\|_{L_t^{2+} L_x^{\infty-}(J_k\times\Omega)} \lesssim 1$, $\|u\|_{L_{t,x}^6(J_k\times\Omega)} \gtrsim 1$,

$$\|u\|_{L_t^\infty L_x^4(J_k\times\Omega)} \gtrsim 1. \tag{4.55}$$

Therefore there exists $t_k \in J_k$ such that $\|u(t_k)\|_{L_x^4(\Omega)} \gtrsim 1$. Moreover, by theorem 2.6 and Sobolev embedding,

$$\|u_{\frac{1}{C(\eta)}N_k \leq \cdot \leq C(\eta)N_k}(t_k)\|_{L_x^4(\Omega)} \gtrsim 1. \tag{4.56}$$

Take K_0 sufficiently large so that $K_0^{-1/3} << \frac{1}{C(\eta)}$.

$$\frac{d}{dt}\int_\Omega |u_{\frac{1}{C(\eta)}N_k \leq \cdot \leq C(\eta)N_k}(t,x)|^4 dx \lesssim C(\eta)^2 N_k^2. \tag{4.57}$$

Therefore, for some $\delta > 0$, $\|u_h(t)\|_{L_x^4(\Omega)} \gtrsim 1$ on $[t_k - \frac{\delta}{C(\eta)^2 N_k^2}, t_k + \frac{\delta}{C(\eta)^2 N_k^2}]$. However this implies

$$\|u_h\|_{L_{t,x}^4(J_k \times \Omega)}^4 \gtrsim \frac{1}{C(\eta)^2 N_k^2}, \tag{4.58}$$

which contradicts (4.54). $\Box$

Theorem 3.1 combined with theorem 4.1 proves theorem 2.7. It only remains to prove (4.5).

5 Endpoint argument

It only remains to prove (4.5). To do this we will upgrade lemma 3.3 to involve l^2 summation. For $K = \sum_{J_k \subset J} \frac{1}{N_k^3}$ we define the norm

$$\|u\|_{X(J\times\Omega)}^2 \equiv \sum_{K^{-1/3} \leq N_j} \frac{1}{KN_j} \sum_{J_{N_j}^l \subset J} \|u_{N_j}\|_{U_\Delta^2(J_{N_j}^l \times \Omega)}^2. \tag{5.1}$$

Theorem 5.1 *If u is a solution to* (1.1), *$\|u\|_{L_{t,x}^6(J\times\Omega)} = M$ for some M sufficiently large and fixed, $E(u(t)) = E_0$, then*

$$\|u\|_{X(J\times\Omega)} \lesssim 1. \tag{5.2}$$

Proof: We again take (3.9). First consider the bad intervals $J_{N_j,b}^l$. By lemma 3.3,

$$\sum_{K^{-1/3} \leq N_j} \frac{1}{KN_j} \sum_{J_{N_j,b}^l} \|P_{N_j} u\|_{U_\Delta^2(J_{N_j,b}^l \times \Omega)}^2 \lesssim \frac{1}{K} \sum_{J_k \subset J} \frac{1}{c^3 N_k^3} \lesssim \frac{1}{c^3}. \tag{5.3}$$

Now turn to the good intervals.

$$\|\Delta P_{N_j} F(u_{\leq N_j})\|_{DU_\Delta^2(J_{N_j,g}^l \times \Omega)} \lesssim \|\Delta u_{\leq N_j}\|_{L_t^{2+} L_x^{4-}(J_{N_j,g}^l \times \Omega)} \|\nabla u_{\leq N_j}\|_{L_t^{2+} L_x^{4-}} \|u_{\leq N_j}\|_{L_t^\infty \dot{H}_x^1} \tag{5.4}$$

$$\lesssim \eta \|\Delta u_{\leq N_j}\|_{U^2_\Delta(J^l_{N_j,g}\times\Omega)}. \tag{5.5}$$

The last inequality follows from lemma 3.3. Now we use the fact that an interval J^l_N, $N \leq N_j$ overlaps $\lesssim (\frac{N_j}{N})^3$ intervals $J^l_{N_j,g}$.

$$\frac{1}{K}\sum_{K^{-1/3}\leq N_j}\frac{1}{N_j}\sum_{J^l_{N_j,g}\subset J}\|P_{N_j}F(u_{\leq N_j})\|^2_{DU^2_\Delta(J^l_{N_j,g}\times\Omega)} \tag{5.6}$$

$$\lesssim \frac{\eta}{K}\sum_{K^{-1/3}\leq N_j}\sum_{K^{-1/3}\leq N\leq N_j}\frac{N^4}{N_j^5}(\frac{N_j}{N})^3\sum_{J^l_N\subset J}\|u_N\|^2_{U^2_\Delta(J^l_N\times\Omega)} \tag{5.7}$$

$$+\eta\sum_{K^{-1/3}\leq N_j}\frac{K^{-2/3}}{N_j^2}\|\nabla u_{\leq K^{-1/3}}\|^2_{U^2_\Delta(J\times\Omega)} \lesssim \eta(1+\|u\|^2_{X(J\times\Omega)}). \tag{5.8}$$

Next,

$$\|P_{N_j}F(u_{N_j<\cdot cN_k})\|_{DU^2_\Delta(J^l_{N_j}\times\Omega)} \lesssim \|P_{N_j}F(u_{N_j<\cdot<cN_k})\|_{L_t^{2-}L_x^{\frac{4}{3}+}(J^l_{N_j,g}\times\Omega)} \tag{5.9}$$

$$\lesssim N_j\sum_{N_j<\cdot M<cN_k}\|u_M\|^{1+}_{L_t^{2+}L_x^{4-}(J^l_{N_j,g}\times\Omega)}\|u_M\|^{1-}_{L_t^\infty L_x^2(J^l_{N_j,g}\times\Omega)}\|u_M\|_{L_t^\infty L_x^4(J^l_{N_j,g}\times\Omega)} \tag{5.10}$$

$$\lesssim \sum_{N_j<M<cN_k}\eta(\frac{N_j}{M})^{1-}\|u_M\|_{U^2_\Delta(J^l_{N_j}\times\Omega)}. \tag{5.11}$$

The last inequality follows from lemma 3.3. Now by taking the convolution of an L^1 function with an L^2 function,

$$\frac{\eta^{1-}}{K}\sum_{K^{-1/3}\leq N_j}\sum_{J^l_{N_j,g}\subset J}(\sum_{N_j<M<cN_k}\frac{1}{N_j^{1/2}}(\frac{N_j}{M})^{1-}(\sum_{J^{l'}_M\cap J^l_{N_j,g}\neq\emptyset}\|u_M\|^2_{U^2_\Delta(J^{l'}_M\times\Omega)})^{1/2})^2 \tag{5.12}$$

$$\lesssim \frac{\eta^{1-}}{K}\sum_{K^{-1/3}\leq N_j}\sum_{J^l_{N_j,g}\subset J}(\sum_{N_j<M<cN_k}(\frac{N_j}{M})^{\frac{1}{2}-}(\sum_{J^{l'}_M\cap J^l_{N_j,g}}\frac{1}{M}\|u_M\|^2_{U^2_\Delta(J^{l'}_M\times\Omega)})^{1/2})^2 \tag{5.13}$$

$$\lesssim \frac{\eta^{1-}}{K}\sum_{K^{-1/3}\leq M}(\sum_{J^{l'}_M\subset J}\frac{1}{M}\|u_M\|^2_{U^2_\Delta(J^{l'}_M\times\Omega)}) \lesssim \eta^{1-}\|u\|^2_{X(J\times\Omega)}. \tag{5.14}$$

Finally,

$$\|\nabla P_{N_j}F(u_{>cN_k})\|_{DU^2_\Delta(J^l_{N_j,g}\times\Omega)} \lesssim \|\nabla P_{N_j}F(u_{>cN_k})\|_{L_t^{2-}L_x^{\frac{4}{3}+}(J^l_{N_j,g}\times\Omega)}. \tag{5.15}$$

By lemma 3.3,

$$\|\nabla P_{N_j}F(u_{>cN_k})\|_{L_t^{2-}L_x^{\frac{4}{3}+}(J^l_{N_j,g}\times\Omega)} \lesssim 1. \tag{5.16}$$

Therefore

$$\sum_{K^{-1/3}\leq N_j}\frac{1}{KN_j^3}\sum_{J^l_{N_j,g}\subset J}\|\nabla P_{N_j}F(u_{>cN_k})\|^2_{L_t^{2-}L_x^{\frac{4}{3}+}(J_{N_j,g}\times\Omega)} \tag{5.17}$$

$$\lesssim \sum_{K^{-1/3}\leq N_j}\frac{1}{KN_j^3}\sum_{J_k\subset J:2N_j\leq N_k}(N_j^2\frac{1}{(cN_k)^2)^{2-}} \tag{5.18}$$

$$\lesssim \frac{1}{K}\sum_{J_k}\frac{1}{(cN_k)^{4-}}\sum_{N_j\leq\frac{N_k}{2}}N_j^{1-}\lesssim\frac{1}{c^{4-}}. \tag{5.19}$$

Finally consider the term

$$\|P_{N_j}((u_{<N_j})(u^2_{>N_j})+(u^2_{<N_j})(u_{>N_j}))\|_{DU^2_\Delta(J^l_{N_j,g}\times\Omega)}. \tag{5.20}$$

First take $(u^2_{<N_j})u_{>N_j}$. Suppose $v=[\Psi(\frac{1}{4}N_j^{-2}\Delta_D)+\Psi(N_j^{-2}\Delta_D)+\Psi(4N_j^{-2}\Delta_D)]v$, $\|v\|_{V^2_\Delta(J^l_{N_j,g}\times\Omega}=1$.

$$\|(u_N)v(u_{<N_j})(u_{>N_j})\|_{L^1_{t,x}(J^l_{N_j,g}\times\Omega)}\lesssim\|v(u_N)\|_{L^2_{t,x}}\|u_{<N_j}\|_{L_t^{2+}L_x^{\infty-}}\|u_{>N_j}\|_{L_t^{\infty-}L_x^{2+}}. \tag{5.21}$$

By lemma 3.3, theorem 3.2,

$$\|u_{>N_j}\|_{L_t^4L_x^{8/3}}\lesssim\frac{1}{N_j}, \tag{5.22}$$

$$\|u_{<N_j}\|_{L_t^4L_x^8}\lesssim\eta^{1/2}. \tag{5.23}$$

$$\|(u_N)v\|_{L^2_{t,x}}\lesssim\|(u_N)v\|^{1-}_{L_t^{2+}L_x^{2-}}\|u_N\|^{+}_{L_t^{2+}L_x^{\infty-}}\|v\|^{+}_{L_t^{2+}L_x^{4-}}\lesssim\frac{N^{3/2-}}{N_j^{1/2-}}\|u_N\|_{U^2_\Delta(J^l_{N_j,g}\times\Omega)}. \tag{5.24}$$

The last inequality follows from interpolating theorem 1.5 with theorem2.6, $V^2_\Delta\subset U^p_\Delta$ for $p>2$.

$$\frac{\eta^{1/2}}{K}\sum_{K^{-1/3}\leq N_j}\frac{1}{N_j}\sum_{J^l_{N_j,g}\subset J}\sum_{K^{-1/3}\leq N\leq N_j}(\frac{N}{N_j})^{3-}\|u_N\|^2_{U^2_\Delta(J^l_{N_j,g}\times\Omega)}\lesssim \eta^{1/2}\|u\|^2_{X(J\times\Omega)}. \tag{5.25}$$

By lemma 3.3,

$$\frac{\eta^{1/2}}{K}\sum_{K^{-1/3}\leq N_j}\frac{1}{N_j}KN_j^3\frac{K^{-1/3}}{N_j^3}\|\nabla u_{K^{-1/3}}\|^2_{U^2_\Delta(J\times\Omega)}\lesssim 1. \tag{5.26}$$

Finally,

$$\|(u_{>N_j})^2(u_{<N_j})\|_{L_t^{2-}L_x^{4/3+}(J^l_{N_j,g}\times\Omega)}\lesssim N_j\|u_{<N_j}\|_{L_t^{\infty-}L_x^{4+}}\sum_{M\geq N_j}\|u_M\|_{L_t^{2+}L_x^{4-}}\|u_M\|_{L_t^\infty L_x^2} \tag{5.27}$$

$$\lesssim \eta^{1-}\sum_{M\geq N_j}\frac{N_j}{M}\|u_M\|_{U^2_\Delta(J^l_{N_j,g}\times\Omega)}. \tag{5.28}$$

$$\frac{\eta^{1-}}{K}\sum_{K^{-1/3}\leq N_j}\frac{1}{N_j}\sum_{J^l_{N_j,g}\subset J}(\sum_{M\geq N_j}\frac{N_j}{M}\|u_M\|_{U^2_\Delta(J^l_{N_j,g}\times\Omega)})^2\lesssim \eta^{1-}\|u\|^2_{X(J\times\Omega)}. \tag{5.29}$$

Finally, for (3.9) choose $t^l_{N_j}\in J^l_{N_j,g}$ such that

$$\|\nabla P_{N_j}u(t^l_{N_j})\|_{L^2_x(\Omega)}=\inf_{t\in J^l_{N_j,g}}(\|\nabla P_{N_j}u(t)\|_{L^2_x(\Omega)}). \tag{5.30}$$

This implies

$$\sum_{K^{-1/3}\leq N_j}\frac{1}{KN_j}\sum_{J^l_{N_j,g}\subset J}\|P_{N_j}u(t^l_{N_j})\|^2_{L^2_x(\Omega)}\lesssim 1. \tag{5.31}$$

Therefore, we have proved

$$\|u\|^2_{X(J\times\Omega)}\lesssim 1+\eta^{1/2}\|u\|^2_{X(J\times\Omega)}. \tag{5.32}$$

This completes the proof of theorem 5.1. $\square$

By lemma 3.3,

$$\|u_N\|_{L_t^{2+}L_x^{4-}(J\times\Omega)}\lesssim(\sum_{J^l_N\subset J}\|u_N\|^{2+}_{L_t^{2+}L_x^{4-}(J^l_N\times\Omega)})^{1/2+}\lesssim N^{-\frac{1}{2}+\frac{1}{2+}}(\sum_{J^l_N\subset J}\|u_N\|^2_{U^2_\Delta(J^l_N\times\Omega)})^{1/(2+)}. \tag{5.33}$$

Also,

$$\|u_N\|_{L_t^{\infty-}L_x^{2+}(J\times\Omega)} \lesssim M^{\frac{1}{\infty-}-1}(\sum_{J_N^l\subset J}\|u_N\|^2_{U^2_\Delta(J_N^l\times\Omega)})^{1/(\infty-)} \tag{5.34}$$

Therefore,

$$\sum_{K^{-1/3}\leq N_1\leq N_2\leq N_3}\|u_{N_1}\|_{L_t^{2+}L_x^{4-}(J\times\Omega)}\|u_{N_2}\|_{L_t^{2+}L_x^{4-}(J\times\Omega)}\|u_{N_3}\|_{L_t^{\infty-}L_x^{2+}(J\times\Omega)} \tag{5.35}$$

$$\begin{aligned}\lesssim &\sum_{K^{-1/3}\leq N_1\leq N_2\leq N_3} N_1^{-1/2+\frac{2}{2+}}N_2^{-1/2+\frac{2}{2+}}N_3^{\frac{2}{\infty-}-1}(\sum_{J_{N_3}^l\subset J}\|u_{N_3}\|^2_{U^2_\Delta(J_{N_3}^l\times\Omega)})^{1/(\infty-)}\\ &\times(\sum_{J_{N_1}^l\subset J}\|u_{N_1}\|^2_{U^2_\Delta(J_{N_1}^l\times\Omega)})^{1/(2+)}(\sum_{J_{N_2}^l\subset J}\|u_{N_2}\|^2_{U^2_\Delta(J_{N_2}^l\times\Omega)})^{1/(2+)}\lesssim\|u\|^2_{X(J\times\Omega)}.\end{aligned} \tag{5.36}$$

The proof of theorem 1.1 is now complete.

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